Last time:

* Magnetic vector potential

$$\underline{B} = \nabla \times \underline{A} \quad \text{where } \underline{A} \equiv \text{magnetic vector potential}$$

$$\nabla \cdot \underline{A} = 0$$

* $\underline{A}$ also fulfills Poisson's eq:

$$\nabla^2 A = -\mu_0 \underline{J}$$

the solutions to this equations:

$$A(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r})\, dV}{r} \quad \text{volume current density}$$

$$A(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{K}(\underline{r})\, da}{r} \quad \text{surface current density}$$

$$A(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{I(r)\, dl}{r} \quad \text{line currents}$$

$\underline{r} \equiv$ vector from the source to the point we are calculating the potential.

* For a magnetic dipole:

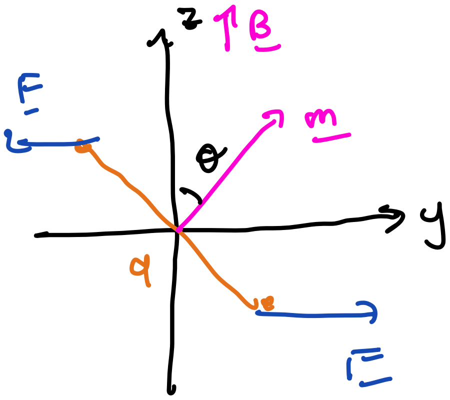
$$\underline{A}_{dip}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{r}}{r^2} \; ; \text{ where } \underline{m} \equiv I \underline{a} \text{ magnetic dipole moment}$$

Diamagnets, paramagnets and ferromagnets

* Paramagnets: Materials that acquire magnetization parallel to the externally applied $\underline{B}$. Materials are attracted towards the regions of stronger field.

* Diamagnets: Materials that acquire magnetization opposite to the external field $\underline{B}$. Relatively weak effect. Materials are mildly repelled by the external $\underline{B}$.

* Ferromagnets: Magnetization is determined by the history. They have a magnetization & retain it independently of the external $\underline{B}$.



The magnitude of the forces on the "tilted" sides of the loop:

$$\underline{F} = I(\underline{l} \times \underline{B})$$

$$F = I b B$$

The torque due to my force is:

$$\underline{\tau} = \underline{r} \times \underline{F}$$

For this force

$$\tau = a F \sin\theta \, \hat{x} = \underbrace{I a b}_{\equiv m} B \sin\theta \, \hat{x}$$

$\equiv m$ magnetic dipole moment

We can write the torque:

$$\boxed{\begin{array}{l} \underline{\tau} = \underline{m} \times \underline{B} \\ \text{where } \underline{m} = I\underline{a} \text{ magnetic dipole moment} \end{array}}$$

This equation has the same form for the electrical analogue.

$\underline{\tau} = \underline{p} \times \underline{E}$ where $\underline{p}$ is the electric dipole

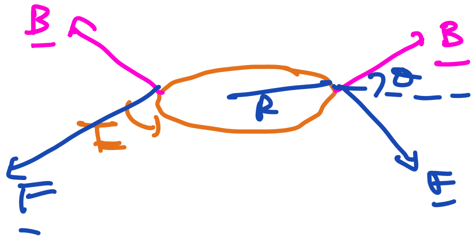
This torque is in a direction that aligns the dipole parallel to the external field. This is the microscopic origin of paramagnetism.

We consider every $\bar{e}$ in the material as a spinning sphere of charge that contributes to the magnetic dipole.

From lecture 13, we saw that the net force on a current loop is zero.

$$\underline{F} = I \oint d\underline{l} \times \underline{B} = I \underbrace{\left[\oint d\underline{l}\right]}_{=0} \times \underline{B} = 0$$

This will no longer be true for a non-uniform $\underline{B}$



$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$$

Downward force on loop due to the radial component of $\underline{B}$

$$F = 2\pi I R B \cos\theta$$

In general for a loop with dipole moment $\underline{m}$, in a field $\underline{B}$:

$$\boxed{\underline{F} = \nabla(\underline{m} \cdot \underline{B})} \qquad \text{If } m \perp \underline{B} \Rightarrow \underline{F} = 0$$

Origin of diamagnetism

The effect $\underline{B}$ on atomic orbits

For a circular orbit

$$T = \frac{2\pi R}{V}$$ period of an orbit
amount of time it takes to complete 1 revolution.

$$I = \frac{q}{t} = \frac{-e}{t} = \frac{-e}{T} = \frac{-eV}{2\pi R}$$ here $-e$ is the charge of the electron.

What is the magnetic dipole moment that corresponds to this current.

$$\underline{m} = I\underline{a} = \left(\frac{-eV}{2\pi R}\right)(\pi R^2)\vec{z} = -\frac{1}{2} eVR\vec{z}$$

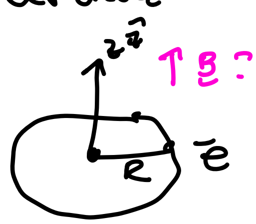
In terms of the orbital angular momentum:

$L = m_e V R$ where m_e is the mass of $\bar{e}$

$$\underline{m} = \frac{-e}{2m_e}\underline{L}$$ where $\underline{L}$ is the orbital angular momentum.

In the absence of an external $\underline{B}$, the electron orbit has a centripetal acceleration v^2/R due to electrical forces alone

$$\underbrace{\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2}}_{F} = \underbrace{m_e\frac{V_0^2}{R}}_{ma}$$



Once we have an external $\underline{B}$, the $\bar{e}$ will experience a force:
$F = -e(\underline{v} \times \underline{B})$ due to $\underline{B}$.

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + evB = \frac{m_e V^2}{R}$$

To determine the change in mag moment we need the change in speed due $\underline{B}$: $\Delta v = v - v_0$

$$evB = m_e \frac{v^2}{R} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = \frac{m_e}{R}(v^2 - v_0^2) = \frac{m_e}{R}(v+v_0)\underbrace{(v-v_0)}_{\Delta v}$$

$$\Rightarrow \Delta v = \frac{eBR}{2m_e}$$

assuming $v \sim v_0$ so $(v+v_0) \sim 2v$

With this we calculate the change in magnetic dipole moment:

$$\Delta \underline{m} = -\frac{1}{2} e (\Delta v) R \hat{z} = -\frac{e^2 R^2 \underline{B}}{4 m_e}$$

change in dipole moment is antiparallel to $\underline{B}$

$\Rightarrow$ origin of diamagnetism

Magnetization

The presence of an external field $\underline{B}$ induces alignment of dipoles that give rise to magnetic polarization

paramagnetism $\longrightarrow$ $\bar{e}$ spins give rise to dipole moments in the direction of $\underline{B}$

diamagnetism $\rightarrow$ $\bar{e}$ orbiting give rise to dipole moments in the opposite direction of $\underline{B}$

ferromagnets $\Rightarrow$ permanent magnetization $\rightarrow$ next class

$M \equiv$ magnetization $\equiv \frac{\text{magnetic dipole moment}}{\text{unit volume}}$

$$\boxed{\underline{M} = \frac{1}{V}\sum_i m_i}$$ where V is the volume

The field of a magnetized material

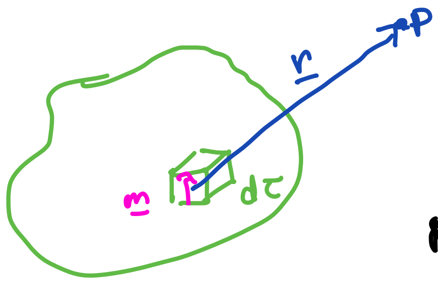
We have a piece of magnetized material, we the magnetic dipole moment / unit volume.

What is the field produced by the material?

For a single magnetic dipole $\underline{m}$. The magnetic vector potential is:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{r}}{r^2}$$

Let's consider the following piece of magnetized material:



by definition of magnetization, each volume element $d\tau$ carries a dipole moment $\underline{M}\, d\tau$

So integrating we obtain the total vector potential:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{M}(\underline{r}) \times \hat{r}}{r^2} d\tau$$

as in the electrical case, we will rewrite this using $\nabla \frac{1}{r} = \frac{\hat{r}}{r^2}$

$$\Rightarrow \underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \underbrace{\left[\underline{M}(\underline{r}) \times \left(\nabla \frac{1}{r} \right) \right]}_{\text{using product rule (v) Griffiths p.21}} d\tau$$

$$\nabla \times \left(\frac{1}{r} \underline{M} \right) = \frac{1}{r} (\nabla \times \underline{M}) - \underline{M} \times \left(\nabla \frac{1}{r} \right)$$

$$\Rightarrow \underline{M} \times \left(\nabla \frac{1}{r} \right) = -\nabla \times \left(\frac{1}{r} \underline{M} \right) + \frac{1}{r} (\nabla \times \underline{M})$$

$$A(\underline{r}) = \frac{\mu_0}{4\pi} \int \left[-\nabla \times \left(\frac{1}{r} \underline{M} \right) + \frac{1}{r} (\nabla \times \underline{M}) \right] d\tau$$

And using the divergence theorem to swap the 1st volume $\int$ for a surface one:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \oint \frac{1}{r} [\underline{M} \times d\underline{a}] + \frac{\mu_0}{4\pi} \int \frac{1}{r} [\nabla \times \underline{M}] d\tau$$

look like the potential due to a surface current

looks like the potential of a volume current

We can rewrite the expression for $\underline{A}$ as:

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\oint_S \frac{\underline{K}_b(\underline{r})\,da}{r} + \frac{\mu_0}{4\pi}\int_V \frac{\underline{J}_b}{r}\,d\tau$$

where $\underline{K}_b = \underline{M}\times\hat{n}$ bound surface current

$\underline{J}_b = \nabla\times\underline{M}$ bound volume current

These are the analogues of the bound volume and surface charge:

$$\rho_b = -\nabla\cdot\underline{P}$$

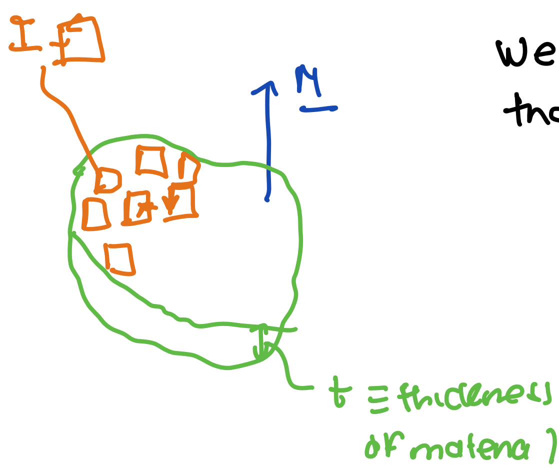
$$\sigma_b = \underline{P}\cdot\hat{n}$$

The vector potential of a magnetized object is the same as the one produced by a volume current density + surface current density.

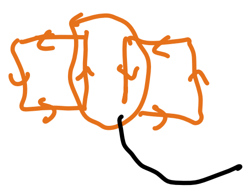
As in the case for dielectrics all problems can be solved by finding these bound currents and calculating the fields generated by them.

Physical interpretation of the bound currents.

Let's consider a slab of uniformly magnetized material

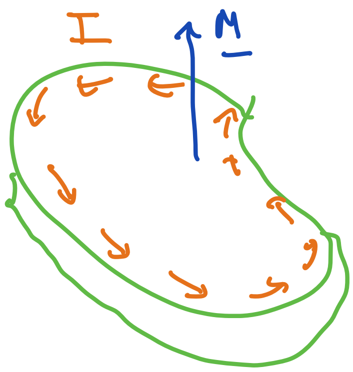


We imagine the material full of "tiny" loops that each produce a dipole moment.

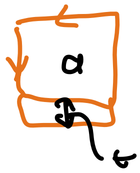


current in adjacent loops will cancel out.

So we will get some effective current



What is the current in terms of $\underline{M}$?



Let's assume each loop has an area a and a thickness t.

In terms of magnetization the dipole is:

$$M = \frac{m}{V} \quad \begin{matrix} \leftarrow \text{dipole moment} \\ \leftarrow \text{volume} \end{matrix}$$

In our case

$$\Rightarrow m = M\underbrace{(a\,t)}_{\text{area x thickness}}$$

In terms of the circulating current the dipole moment is:

$$m = Ia$$

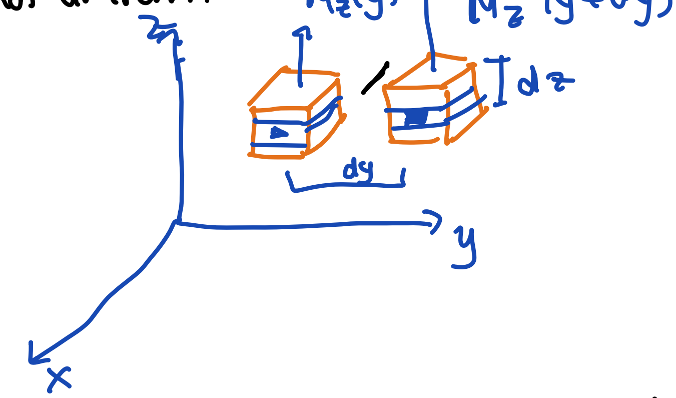
Setting the expressions for m equal to each other

$$Mat = Ia \Rightarrow \frac{I}{t} = M \Rightarrow K_b = M$$

$\equiv K_b$ current/area $\underline{M} \times \hat{n}$ is the direction of K_b

$\therefore \underline{K}_b = \underline{M} \times \hat{n}$ because $\underline{M} \parallel \hat{n}$ there is no current on top or bottom of slab since x product vanishes.

What happens if $\underline{M}$ is not uniform



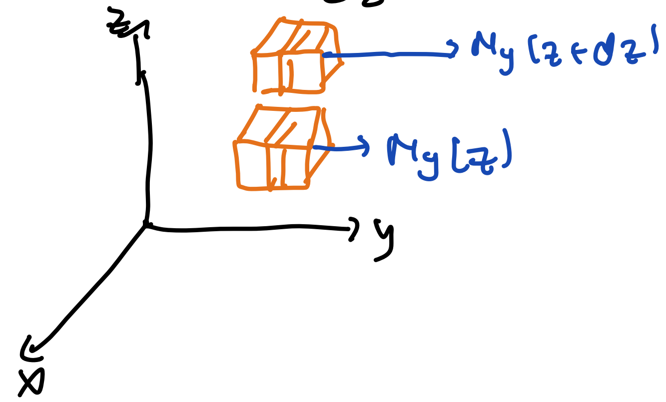
If we have a large magnetization at $y+dy$ compared to y we get a current in the x direction.

$$I_x = [M_z(y+dy) - M_z(y)]\,dz$$

$$= \frac{\partial M_z}{\partial y} dz\, dy$$

$$\Rightarrow \underbrace{\frac{I_x}{dz\,dy}}_{\equiv J_b} = \frac{\partial M_z}{\partial y} \Rightarrow J_{b_x} = \frac{\partial M_z}{\partial y}$$

Considering a non uniform current in the z axis would contribute $-\frac{\partial M_y}{\partial z}$



In general considering both contributions

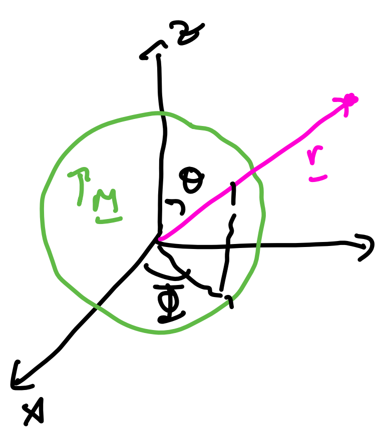
$$J_{bx} = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

In generally

$$\underline{J}_b = \nabla \times \underline{M}$$

Non uniform $\underline{M}$ gives rise to a bound volume current.

Example: Find the magnetic field of a uniformly magnetized sphere.



we choose a coord system where the z axis is aligned with the magnetization $\underline{M}$.

The goal is to find the bound currents.

$\rightarrow \underline{K}_b = \underline{M} \times \hat{n}$

$\underline{J}_b = \nabla \times \underline{M} = 0$ b/c we have a uniform magnetization.

All we have left to do is calculate $\underline{K}_b = \underline{M} \times \hat{n}$

We chose $\underline{M}$ // to $\hat{k}$ so

$$\underline{M} = M\hat{k}$$

In our case $\hat{n}$ is a radial vector so $\hat{n} = \hat{r}$. We need to compute

$$\underline{M} \times \hat{n} = M\hat{k} \times \hat{r}$$

We'll choose to write $\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

$$\underline{M} \times \hat{n} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\Phi} \\ M\cos\theta & -M\sin\theta & 0 \\ 1 & 0 & 0 \end{vmatrix} = M\sin\theta\,\hat{\Phi}$$

$$\boxed{\hat{e}_z = \hat{k} = \hat{z}}$$

$$\Rightarrow K_b = M\sin\theta\,\hat{\Phi}$$

We consider a sphere with a uniform surface charge σ, rotating w/ angular vel $\underline{\omega}$:

$$\underline{K}_b = \sigma\underline{v} = \sigma\omega R\sin\theta\,\hat{\phi}$$

We have:

$$\underline{B} = \frac{2}{3}\mu_0\sigma R\underline{\omega} =$$